

Tropospheric and Ionospheric Range Corrections for an Arbitrary Inhomogeneous Atmosphere (First-Order Theory)

O. H. von Roos

Tracking and Orbit Determination Section

In this article a simple and concise expression for the range correction for an atmosphere which possesses arbitrary radial, lateral, and azimuthal gradients of the index of refraction is presented. The validity of this expression hinges only on the assumption that the index of refraction is close to unity, an assumption which is well satisfied for the Earth's atmosphere. Furthermore, it is shown that the range corrections for a simple model of the Earth's troposphere, including typical lateral variations, are in close agreement with existing computer solutions.

I. Introduction

Hand in hand with the ever increasing complexity of unmanned space missions, there is an ever increasing need for higher accuracies in orbit determinations. The orbit or trajectory is primarily determined by measuring the range, ρ , and by measuring the time rate of change of the range, $\dot{\rho}$. The present status of the accuracy is about 7 m for range and 1.3 m/12 h for the range rate (Ref. 1). That is, at a distance of 1 AU ($1.5 \cdot 10^8$ km) the range can be measured within 1 m (approximately 1 part in 10^{11}), and the velocity of the spacecraft (typically 10 km/s) is known to 1 part in 10^9 . In order to achieve these enormous accuracies, a great number of

error sources have to be analyzed and accounted for. Since the range is measured by a time measurement (the transit time of a radio signal between Earth and the spacecraft) and velocities are measured by cycle counting (doppler shift), there ensues a plethora of possible error sources. Those errors due to hardware and software of the DSN have been enunciated elsewhere (Ref. 2). Further, atmospheric and cosmic noise is, of course, degrading the signal. But this is not all. Severe degradation of the range determination comes about via the interaction of the radio signal with the intervening tenuous matter. These interactions can conveniently be classified as due to the troposphere, the ionosphere, and the solar wind. Although the refractive index of these strata

is close to unity, they nevertheless give rise to sizable range errors, sizable in the sense that the errors introduced are larger than the required accuracy (1 m). For instance, at an elevation angle of 10 deg in a configuration such that the spacecraft is at a distance of 1 AU and the Sun-Earth-probe angle is 40 deg and it happens to be local noon (the ionospheric electron concentration is higher in daytime), the range error due to the troposphere is 10 m, that due to the ionosphere is 10 m, and, finally, that due to the solar wind is 10 m, and *they are all additive*. Of course, the values just quoted are only representative and are by no means accurate, since all so-called "media range corrections" are sensitive functions of their respective variables (elevation angle for instance, etc.).

In this article, we address ourselves to two main culprits: the troposphere and the ionosphere or, generally speaking, the atmosphere. It will be shown that the range correction can actually be determined by a simple and concise expression (Eq. 42 of the text).¹ There is no need for extensive ray-tracing programs. All that has to be known is the refractive index profile within the unperturbed (straight) ray path. The accuracy of expression (42), which makes use of the fact that $n - 1 \ll 1$, where n is the refractive index, is proportional to $(n - 1)^2$ and therefore lies well within the cm range. It is hoped that the simplification arrived at in the following pages alleviates to some extent the complex question of range calibration.

II. Troposphere

In the spherical coordinate system depicted in Fig. 1, we express the refractive index of the troposphere by

$$n(r, \theta) = 1 + \alpha F(r, \theta) \quad (1)$$

where $\alpha \ll 1$ and F is an arbitrary function of the two variables, r the distance from the Earth's center, and θ the angle between an (arbitrary) z axis and the radial distance to a point in space. A fairly accurate example is given by the following very simple model of the troposphere:

$$\alpha = 3 \cdot 10^{-4} \left\{ F(r, \theta) = \exp \left[-\frac{r - R}{H} \right] \right\} \quad (2)$$

¹Knowing, of course, the intervening density profiles of both the troposphere and the ionosphere.

an exponentially decreasing refractive index with scale height H . As we can see, the requirement $\alpha \ll 1$, which will play a vital role in our subsequent analysis, is indeed satisfied.

Let us suppose for a moment that $\alpha = 0$, or that the atmosphere is absent. In this case the ray path between a distant spacecraft and the point of observation, specified by the coordinates $r = R$ and $\theta = \bar{\theta}$, is a straight line. A convenient expression is

$$\theta(r) = \theta_0(r) = \bar{\theta} - \gamma + \cos^{-1} \left(\frac{R \cos \gamma}{r} \right) \quad (3)$$

that is to say that each point on the straight line, expressed analytically by Eq. (3), is uniquely determined by its geocentric distance r . From Eq. (3), it follows that

$$\frac{r^2 \theta'_0}{\sqrt{1 + r^2 (\theta'_0)^2}} = R \cos \gamma \quad (4)$$

which we need for future references.²

Suppose now that the troposphere is present and represented by the refractive index (1). Taking r as the independent variable, the ray path may now be expressed by³

$$\theta(r) = \theta_0(r) + \theta_1(r) \quad (5)$$

where θ_0 is given by Eq. (3) and

$$\frac{\theta_1}{\theta_0} \ll 1 \quad (6)$$

due to the fact that $n - 1 \ll 1$. We do not need to solve the Euler equation associated with θ_1 . All we have to do to determine θ_1 uniquely is to postulate that at the observation point:

$$\left. \begin{aligned} \theta_1(R) &= 0 \\ \theta'_1(R) &= 0 \end{aligned} \right\} \quad (7)$$

This is not a restriction since the differential equation governing θ_1 is of second order. Taking the elevation angle explicitly into account, we may write for the ray path disturbed by the troposphere:

$$\theta(r, \gamma) = \theta_0(r, \gamma) + \theta_1(r, \gamma) \quad (8)$$

²The prime means differentiation with respect to r .

³Omitting azimuthal variations for the time being.

From Eq. (3), it is clear that the ray (8) will emerge from the troposphere at a certain angle, which for large r becomes

$$\theta(\infty, \gamma) = \frac{\pi}{2} + \bar{\theta} - \gamma + \theta_1(\infty, \gamma) \quad (9)$$

On the other hand, the unperturbed beam's limiting angle for large r is simply given by

$$\theta_0(\infty, \gamma) = \frac{\pi}{2} + \bar{\theta} - \gamma \quad (10)$$

The reason for this, of course, is that the unperturbed and the perturbed beams start off at the same elevation angle. To remedy the situation we must insist that the two beams are parallel to each other (for a distant spacecraft) after having left the troposphere. The angles (9) and (10) must therefore be the same. To achieve this we must correct the elevation angle γ in Eq. (8) in the following manner:

$$\begin{aligned} \theta(r, \gamma') &= \theta_0(r, \gamma + \theta_1(\infty, \gamma)) + \theta_1(r, \gamma) \\ &= \theta_0(r, \gamma) + \theta_1(\infty, \gamma) \frac{\partial \theta_0}{\partial \gamma} + \theta_1(r, \gamma) \end{aligned} \quad (11)$$

again in first order. From Eq. (3), it follows that

$$\frac{\partial \theta_0}{\partial \gamma} = -1 + \left(1 - \frac{R^2 \cos^2 \gamma}{r^2}\right)^{-1/2} \frac{R}{r} \sin \gamma \quad (12)$$

so that, indeed,

$$\theta(\infty, \gamma') = \theta_0(\infty, \gamma) \quad (13)$$

It is now an easy matter to compute the range correction due to the influence of the troposphere. It is clear that the topocentric distance to the spacecraft is simply given by

$$\rho_0 = \int_R^\infty \sqrt{1 + r^2 [\theta_0(r, \gamma)]^2} dr \quad (14)$$

if the atmosphere is absent. The angle θ_0 in Eq. (14) is given by Eq. (3). Therefore the range Eq. (14) is merely an integral along a straight line. The upper limit ∞ is only taken for convenience, signifying that the spacecraft is many Earth radii away. On the other hand, the ap-

parent topocentric range, taking explicitly the troposphere into account, is given by

$$\rho_1 = \int_R^\infty n(r, \theta(r, \gamma')) \sqrt{1 + r^2 [\theta(r, \gamma')]^2} dr \quad (15)$$

where $\theta(r, \gamma')$ is found from Eq. (11) and therefore duly recognizes the correction to the elevation angle. In first order we have then for the range correction:

$$\begin{aligned} \Delta \rho = \rho_1 - \rho_0 &= \int_R^\infty \frac{r^2 \theta'_0}{\sqrt{1 + r^2 (\theta'_0)^2}} \left(\theta'_1 + \theta_1(\infty) \frac{\partial \theta'_0}{\partial \gamma} \right) dr \\ &+ \int_R^\infty (n - 1) \sqrt{1 + r^2 (\theta'_0)^2} dr \end{aligned} \quad (16)$$

Taking into account Eqs. (4) and (12), we see that the first integral in Eq. (16) actually vanishes since

$$\begin{aligned} \int_R^\infty \frac{r^2 \theta'_0}{\sqrt{1 + r^2 (\theta'_0)^2}} \left(\theta'_1 + \theta_1(\infty) \frac{\partial \theta'_0}{\partial \gamma} \right) dr \\ = R \cos \gamma \theta_1(\infty) \left(1 + \frac{\partial \theta_0(\infty)}{\partial \gamma} \right) = 0 \end{aligned} \quad (17)$$

We have, therefore, the result, using explicitly Eq. (3) for θ_0 :

$$\Delta \rho = \int_R^\infty (n - 1) \left(1 - \frac{R^2 \cos^2 \gamma}{r^2} \right)^{-1/2} dr \quad (18)$$

We have then the remarkable result that as long as $n - 1 \ll 1$ and a first-order perturbation theory is adequate, the range correction is simply the integral over the unperturbed (straight) ray path weighted with that part of the refractive index which deviates from unity. There is no need to compute the actual ray path, the bending and twisting of it and so forth. Inserting Eq. (1) into Eq. (18) yields

$$\Delta \rho = \alpha \int_R^\infty F(r, \theta_0(r, \gamma)) \left(1 - \frac{R^2 \cos^2 \gamma}{r^2} \right)^{-1/2} dr \quad (19)$$

where, of course, θ_0 is given by Eq. (3). Equation (19) holds for all elevation angles but must be computed numerically for most cases of interest, particularly at low elevation angles. However, at higher elevation angles ($\gamma > 10^\circ$), considerable simplifications arise if simple but

adequate tropospheric models are used. We shall do so now. For $\gamma > 10^\circ$ and an exponentially decreasing refractive index, the square root under the integral sign may be approximated by

$$\left(1 - \frac{R^2}{r^2} \cos^2 \gamma\right)^{-1/2} \approx (\sin \gamma)^{-1} \quad (20)$$

and if we assume for $n - 1$:

$$n - 1 = \alpha [1 \pm 2(\theta_0 - \bar{\theta})] \exp\left[-\frac{r - R}{H}\right] \quad (21)$$

with $\alpha = 3 \cdot 10^{-4}$ and $H = 7$ km,⁴ we obtain immediately from Eq. (19)

$$\Delta\rho = \frac{\alpha}{\sin \gamma} H \pm \frac{2\alpha}{\sin \gamma} \int_R^\infty dr \exp\left[-\frac{r - R}{H}\right] (\theta_0(r, \gamma) - \bar{\theta}) \quad (22)$$

The first term on the right-hand side of Eq. (22) represents the contribution to the range correction due to the radially stratified troposphere. The second term is due to lateral inhomogeneities. With the aid of Eq. (3), it can be expressed as

$$\Delta\rho_L = \frac{2\alpha}{\sin \gamma} \int_R^\infty dr \exp\left[-\frac{r - R}{H}\right] \left\{ \cos^{-1}\left(\frac{R \cos \gamma}{r}\right) - \gamma \right\} \quad (23)$$

The rapid decrease of the integrand, because of the exponential, makes it clear that for the curly bracket we can use the following approximation:

$$\begin{aligned} \cos^{-1}\left(\frac{R \cos \gamma}{r}\right) - \gamma &= \left\{ \frac{\partial}{\partial r} \cos^{-1}\left(\frac{R \cos \gamma}{r}\right) \right\}_{r=R} (R - r) \\ &= \frac{\cos \gamma}{\sin \gamma} \frac{r - R}{R} \end{aligned} \quad (24)$$

Using Eq. (24) we obtain readily for the integral (23):

$$\Delta\rho_L = \frac{2\alpha \cos \gamma}{\sin^2 \gamma} \frac{H}{R} H \quad (25)$$

Therefore, the total range correction is given by

$$\Delta\rho = \frac{\alpha}{\sin \gamma} H \pm \frac{2\alpha \cos \gamma}{\sin^2 \gamma} \frac{H}{R} H \quad (26)$$

⁴The value $\pm 2(\theta_0 - \bar{\theta})$ for typical lateral inhomogeneities of the refractive index is a good approximation to existing data (Ref. 3).

for elevation angles γ larger than 10 deg. For elevation angles less than 10 deg approximately, Eq. (19) must be determined numerically. However, determining $\Delta\rho$ from Eq. (26), together with the numerical values given above, yields at $\gamma = 10$ deg:

$$\Delta\rho = (12.1 \pm 0.2) \text{ meter} \quad (27)$$

in complete agreement with the work of other researchers (Ref. 4).⁵

We now extend this work to the case in which n is also a function of the azimuthal angle Φ , in other words $n = n(r, \theta, \Phi)$. It is clear that if $\bar{\theta} = 0$ in Fig. 1, every unperturbed ray lies in a meridian plane and its azimuthal angle is $\Phi_0 = \text{constant}$. Suppose that n is also a function of Φ . In that case the ray path will be twisted ever so slightly, also in an azimuthal direction, and we have

$$\Phi(r) = \Phi_0 + \Phi_1(r, \gamma) \quad (28)$$

where again $\Phi_1/\Phi_0 \ll 0$ and

$$\begin{cases} \Phi_1(R, \gamma) = 0 \\ \Phi'_1(R, \gamma) = 0 \end{cases} \quad (29)$$

just as in Eq. (7). In order for the ray path to be parallel to the unperturbed ray after leaving (or before entering) the troposphere (as the case may be), we must have

$$\Phi'_0 = \Phi_0 + \Phi_1(\infty, \gamma) \quad (30)$$

But since the line element in this case is given by

$$dS = \sqrt{1 + r^2(\theta')^2 + r^2 \sin^2 \theta (\Phi')^2} dr \quad (31)$$

and noting that $(\Phi')^2 = (\Phi'_1)^2$ is of second order, we see that the result (19) is still valid with the stipulation that

$$\begin{aligned} \Delta\rho &= \alpha \int_R^\infty F(r, \theta_0(r, \gamma), \Phi_0) \\ &\times \left(1 - \frac{R^2 \cos^2 \gamma}{r^2}\right)^{-1/2} dr \end{aligned} \quad (32)$$

⁵Also see Chao, C. C., "Tropospheric Range Effect Due to Simulated Inhomogeneities by Ray Tracing" (this volume).

This is true in the special coordinate system in which $\bar{\theta} = 0$. But since $\Delta\rho$ is an invariant length in any general coordinate system, expression (32) is still valid but the angles θ_0 and Φ_0 have to be replaced by their appropriate values. If, for instance, the angle between the projection of the line of sight and the meridian is α and if we put $\Phi_0 = 0$ for simplicity, the connection between the angles θ and Φ of the general coordinate system⁶ and the special angles θ_0 and $\Phi_0 = 0$ is given by

$$\left. \begin{aligned} \sin \theta \sin \Phi &= \sin \alpha \sin \theta_0 \\ \cos \theta &= -\cos \alpha \sin \bar{\theta} \sin \theta_0 + \cos \bar{\theta} \cos \theta \end{aligned} \right\} \quad (33)$$

In any case, it is clear from Eq. (32) that azimuthal gradients only give rise to second-order effects for the range correction.

III. Ionosphere

The work done so far has only dealt with the troposphere, but it is easy enough to incorporate the ionosphere in the same vein. For instance, consider the refractive index of a plasma:

$$n = \sqrt{1 - \frac{w_p^2}{w^2}} \quad (34)$$

where the plasma frequency w_p is given by

$$w_p = \left(\frac{4\pi e^2}{m} N \right)^{1/2} \quad (35)$$

with N the electron number density per cm^3 , and w the frequency of the radio beam traversing the ionosphere. For the DSN frequencies $w \approx 10^{10} \text{ sec}^{-1}$ and for the prevailing ionospheric electron densities of $N \approx 10^6 \text{ cm}^{-3}$, we have

$$n = 1 - \frac{w_p^2}{2w^2} \quad (36)$$

since $w_p^2/w^2 \approx 10^{-5}$ is very small. For a determination of the ray path in the charged-particle medium, essentially

the same Fermi principle holds as it does for the troposphere, namely,

$$\delta \int n(r, \theta, \Phi, w) ds = 0 \quad (37)$$

where n is now given by Eq. (36). To determine the range correction we must, however, be precise because of dispersion. The range is usually measured by comparing a received range code with an internally generated code carefully calibrated with a clock. A range code constitutes a modulated signal and propagates, therefore, with the group velocity which differs from the phase velocity by w_p^2/w^2 within the approximation used to obtain Eq. (36). Whereas the phase velocity is given by

$$v_p = c \left(1 + \frac{w_p^2}{2w^2} \right) \quad (38a)$$

the group velocity is given by

$$v_g = c \left(1 - \frac{w_p^2}{2w^2} \right) \quad (38b)$$

It is also clear that the modulation is "riding" on the carrier frequency or in Fourier language, the modulation consists of frequency components very close to the carrier frequency and therefore the ray path is governed by Eq. (37) with w being the carrier frequency ($\approx 10^{10} \text{ rad/s}$). Accordingly, the range correction due to the ionosphere is determined by the difference

$$\Delta\rho_i = c \int \frac{dS}{v_g} - \rho_0 \quad (39)$$

where dS is a solution of Eq. (37) and ρ_0 is given by Eq. (14). The same analysis which led to Eq. (18) can now be carried out without change to yield the following expression for the range correction:

$$\Delta\rho_i = \int_R^\infty \frac{w_p^2}{2w^2} \left(1 - \frac{R^2 \cos^2 \gamma}{r^2} \right)^{-1/2} dr \quad (40)$$

Furthermore, it can be shown from Maxwell's equations (Ref. 5) that the dielectric constant of a medium consisting of a neutral polarizable background (here the troposphere) and a plasma (here the ionosphere) is simply given by the sum:

$$\epsilon = \epsilon_0 - \frac{w_p^2}{w^2} \quad (41)$$

⁶A coordinate system which may have the Earth's spin axis as Z axis. We are thinking of a global chart of tropospheric conditions, something which does not exist yet.

where ϵ_0 is the dielectric constant of the neutral medium. Collecting Eqs. (32), (35), and (40) and expressing everything in terms of the unperturbed ray path, we have as a final result:

$$\Delta\rho = \int_R^\infty dr \left(1 - \frac{R^2 \cos^2 \gamma}{r^2} \right)^{-1/2} \left\{ \alpha F(r, \theta_0(r, \gamma), \Phi_0) + \frac{2\pi e^2}{mw^2} N(r, \theta_0(r, \gamma), \Phi_0) \right\} \quad (42)$$

expressing the range correction as a single integral over the unperturbed ray path, combining both the troposphere and the ionosphere in one expression.

In conclusion we would like to mention and elaborate on the time variations of the atmosphere. First of all, it is clear that any time variations (day-night changes, seasonal changes, turbulent fluctuations, etc.) by and large have a very long time scale compared to the transit time of the signal which is of the order of 10^{-2} s at most. Therefore, time enters Eq. (42) strictly as an additional parameter t . The range rate or the time derivative of the range is then given by expression (42), in which both F and N are differentiated with respect to time. If, however, the range rate $\Delta\dot{\rho}_D$ is measured via doppler by referencing cycles of a received monochromatic signal with an internally generated monochromatic signal (a

clock), the correct expression for $\Delta\dot{\rho}$, taking the phase velocity Eq. (38a) into account, is

$$\Delta\dot{\rho}_D = \int_R^\infty dr \left(1 - \frac{R^2 \cos^2 \gamma}{r^2} \right)^{-1/2} \left\{ \alpha \frac{d}{dt} F(r, \theta_0(r, \gamma), \Phi_0; t) - \frac{2\pi e^2}{mw^2} \frac{d}{dt} N(r, \theta_0(r, \gamma), \Phi_0; t) \right\} \quad (43)$$

IV. Summary

On the preceding pages, simple expressions, both for the range correction $\Delta\rho$ and for the range rate correction $\Delta\dot{\rho}$ due to a tenuous atmosphere, have been derived. To be sure these expressions are approximate and are only valid if the refractive index of the atmosphere is close to unity. This is very well satisfied for the Earth's atmosphere. In as much as second-order effects may be neglected, the theory presented here is completely general. All that has to be known is the electron density profile and the neutral gas density profile within the undisturbed line of sight between the Earth-bound station and the spacecraft. Although, as already mentioned, the expression for the range correction is rather simple, for low elevation angles and realistic density profiles the integral must probably be computed numerically. Work along these lines and also the incorporation of the solar wind into our model is in progress.

References

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